Lecture 13

Thursday, 8 October 2026 6:03 PM

Last time:

Set N of n voters, A of m alternatives / candidates

Each voter has a strict linear order $\succ_i \in L(A)$. (= preference / ranking)

Profile $P = (\succ_1, \ldots, \succ_n)$

Social Choice Function $f: P \to A$

Correspondence $F: P \to 2^A$

Welfare Function $F: P \to L(A)$

A SCF F is:

(i) Unanimous if: $\forall i\ a \succ_i b \Rightarrow a \succ_P b$

(ii) Independent of Irrelevant Alternatives (IIA) if

Given profiles P & P':

$\forall i\ (a \succ_i b) \iff (a \succ'_i b)$

$\Rightarrow (a \succ_P b) \iff (a \succ'_P b)$

(iii) Dictatorship if $\exists i$: $F(P) = \succ_i \quad \forall P = (\succ_1, \succ_n)$

Arrow's Impossibility Theorem: For $m \geq 3$, any SCF that is unanimous & IIA is a dictatorship.

Today: Gibbard - Satterthwaite, for social choice functions.

We write $P = (\succ_1, \succ_n)$ when we want to separate it's ranking from the others.

An SCF is manipulable by i at P if:

$P = (\succ_1, \succ_n)$, & $\exists \succ'_i \in L(A)$ s.t.

$f(\succ'_i, \succ_{-i}) \succ_i f(P)$

An SCF that is not manipulable by any voter at any profile is called non-manipulable, or truthful or strategy-proof.

An SCF f is onto if $\forall a \in A,\ \exists P : f(P) = a$

An SCF f is a dictatorship if $\exists i\ \forall P$, $f(P)$ = top-ranked candidate of $\succ_i$

G-S Theorem: For $m \geq 3$, any SCF that is onto, & truthful is a dictatorship.

$L(a, \succ) := \{b : a \succ b\}$ is called the lower contour set of a in $\succ$.

For a profile P & P' and $a \in A$, P' is an a-improvement of P if $\forall i$: $L(a, \succ_i) \subseteq L(a, \succ'_i)$.

(in P)

(i.e. $\forall i$, every candidate that was below a in $\succ_i$, remains below)

f is monotone if: Let $f(P) = a$, and P' is a a- improvement of P. Then $f(P') = a$.

Claim: f is truthful $\Rightarrow f$ is monotone

Proof: Suppose f is not monotone. Then $\exists P = (\succ_1, \ldots, \succ_n)$, $F(P) = a$, $P' = (\succ'_1, \ldots, \succ'_n)$ is an a- improvement, & $F(b) \neq a$

Let $P^0 = P = (\succ_1, \ldots, \succ_n)$

$P^1 = (\succ'_1, \succ_2, \ldots, \succ_n)$

$P^k = (\succ_1, \succ'_1, \ldots, \succ'_k, \succ_{k+1}, \ldots, \succ_n)$

$P^n = P' = (\succ'_1, \succ'_1, \ldots, \succ'_n)$.

Now $\exists k \in [n]$ s.t. $f(P^{k-1}) = a$, $f(P^k) = b \neq a$

B/w P^{k-1} & P^k only voter k shifted from $\succ_k$ to $\succ'_k$

Further $L(a, \succ_k) \subseteq L(a, \succ'_k)$.

Now Suppose $b \succ_k a$. Let k be actual profile for $\succ_k$.

Then $f(P^{k-1}) = a$, $f(P^k) = b$, & f is not truthful.

Suppose $a \succ_k b$. Let k be actual profile for $\succ'_k$.

Since P' is an a- improvement, $a \succ'_k b$.

Then $f(P^k) = b$, and $f(P^{k-1}) = a$, & f is not truthful. ∎

Claim: Let f be monotone & onto. Consider P where $\forall i$, rank$_i(a) = \perp$.

Then $f(P) = a$.

Proof: Since f is onto, $\exists P'$ s.t. $f(P') = a$. But note that P is a a- improvement of P'. Hence by monotonicity, $f(P) = a$. ∎

Given P & $B \subseteq A$, we define P^B as the profile where every voter moves B to the top of his preference order, keeping relative order within B unchanged & within $A \setminus B$ unchanged.

i.e. in P^B, $\forall i$, (i) $\forall a, a' \in B$, $a \succ_i a'$ in P^B iff $a \succ_i a'$ in P

(ii) $\forall a, a' \notin B$, $a \succ_i a'$ in P^B iff $a \succ_i a'$ in P

(iii) $\forall a \in B, a' \notin B$, $a \succ_i a'$ in P^B.

Claim: Let f be monotone & onto. Let P be a profile where $\forall i$, $a \succ_i b$.

Then $f(P) \neq b$.

Proof: Consider the profile $P^{\{a,b\}}$. In $\forall i$, $a \& b$ are ranked 1st & 2nd respectively. Then $f(P^{\{a,b\}}) = a$. But $P^{\{a,b\}}$ is a b-improvement over P, so $f(P^{\{a,b\}}) = b$. Contradiction! ∎

Corollary: Let f be monotone & onto. For any P, $B \subseteq A$,

$f(P^B) \in B$

Claim: Let f be monotone, $S \subseteq T \subseteq A$, $a \in S$

For a profile P if $f(P^T) = a$ then $f(P^S) = a$.

(Easy since P^S is an a- improvement over P^T).

Steps in G-S theorem proof:

① Use f (monotone, onto) to create a SWF F

② Show monotone + onto $\Rightarrow$ unanimity & IIA for F

③ Show that dictator in $F \Rightarrow$ dictator in f.

Step 1:-

Given P, to obtain $F(P) \in L(A)$:

- $\forall a, b \in A$, consider $f(P^{\{a,b\}}) \in \{a, b\}$

If $f(P^{\{a,b\}}) = a$, then $a \succ b$, else $b \succ a$ for $F(P)$

Need to show acyclicity i.e. $a \succ b$ & $b \succ c$ then $a \succ c$

For a contradiction, suppose $a \succ b$ & $b \succ c$ but $c \succ a$

i.e. $f(P^{\{a,b\}}) = a$, $f(P^{\{b,c\}}) = b$, $f(P^{\{a,c\}}) = c$

Now Consider $f(P^{\{a,b,c\}}) = a$.

By Claim, $f(P^{\{a,b\}}) = f(P^{\{a,c\}}) = a$. Contradiction ∎

Thus, $F(P) \in L(A)$

Step 2: f monotone & onto $\Rightarrow$ F unanimous + IIA.

F is unanimous:

WTS: in P if $\forall i$ $a \succ_i b$ then in F, $a \succ_P b$

$\iff f(P^{\{a,b\}}) = a$. Easy since $f(P^{\{a,b\}})$ has a ranked 1st $\forall i$.

F is IIA:

WTS: P, Q are s.t. each i has the same relative ordering for a, b in P as in Q

Then $a \succ_P b$ iff $a \succ_Q b$

$\iff f(P^{\{a,b\}}) = a \iff f(Q^{\{a,b\}}) = a$

Proof: Say $f(P^{\{a,b\}}) = a$. Now $Q^{\{a,b\}}$ is a a- improvement on $P^{\{a,b\}}$

Hence $f(Q^{\{a,b\}}) = a$. ∎

Thus, $\exists k$: $\forall P$, $F(P) = \succ_k$

Claim: $\forall P$, $f(P)$ = top-ranked candidate of $\succ_k$

Prove yourself

Hence k is a dictator, proving the theorem.